

Rational approximants for the Euler-Gompertz constant*

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Abstract

We obtain two sequences of rational numbers which converge to the Euler-Gompertz constant. Denote by $\langle f(x) \rangle$ the integral of $f(x)e^{-x}$ from 0 to infinity. Recall that the Euler-Gompertz constant δ is $\langle \ln(x+1) \rangle$.

Main idea. Let $P_n(x)$ be a polynomial with integer coefficients. It is easy to prove that $\langle P_n(x) \ln(x+1) \rangle = a_n + \langle \ln(x+1) \rangle b_n$ for some integers a_n, b_n . Hence if $\langle P_n(x) \ln(x+1) \rangle / b_n$ converges to zero, a_n/b_n converges to $-\delta$.

Main Theorem. *Let u be positive real. There exists polynomials $P_n(x)$ (they are explicitly given in the paper) such that $\langle P_n(x) \ln(xu+1) \rangle$ tends to u as n tends to infinity.*

Proof of Main Theorem is elementary.

1 Main result

Theorem 1.1. *For each real $u \geq 0$*

$$u = \sum_{m=r}^{\infty} \sum_{k=r}^m \binom{m}{k} \binom{k}{r} \frac{(-1)^{k+r}}{k!} \int_0^{\infty} x^{k-1} e^{-x} \ln(xu+1) dx.$$

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We recall that

$$\delta = \int_0^{\infty} \ln(x+1)e^{-x} dx.$$

Corollary 1.2. *Let $r \geq 0$ be integer. We define two sequences of integer numbers a_m and b_m by formulas*

$$a_m = \sum_{k=r}^m \binom{m}{k}^2 \binom{k}{r} (m-k)! \sum_{w=0}^{k-1} (-1)^w w! \quad b_m = \sum_{k=r}^m \binom{m}{k}^2 \binom{k}{r} (m-k)!.$$

$$\text{Then } \lim_{m \rightarrow \infty} \frac{a_m}{b_m} = -\delta.$$

Corollary 1.3. *Let $r \geq 1$ be integer. We define two sequences of integer numbers a_m and b_m by formulas*

$$a_m = m! \sum_{k=r}^m \sum_{j=0}^{k-1} \sum_{i=0}^{j-1} \binom{m}{k} \binom{k}{r} \frac{i!}{kj!} (-1)^{k+j+i+1}$$

$$b_m = m! \sum_{k=r}^m \sum_{j=0}^{k-1} \binom{m}{k} \binom{k}{r} \frac{(-1)^{k+j}}{kj!}.$$

$$\text{Then } \lim_{m \rightarrow \infty} \frac{a_m}{b_m} = -\delta.$$

Conjecture 1.4. *For each real $u > 0$*

$$\psi(u) = \ln(u) + \lim_{m \rightarrow \infty} \sum_{k=1}^m A_{k,m+1} \binom{m}{k} \frac{(-1)^k}{k!m!} \int_0^{\infty} x^{k-1} e^{-x} \ln \left(\frac{x+u}{u} \right)$$

where

$$A_{k,m} = \sum_{t=2}^m \left\{ \begin{matrix} m \\ t \end{matrix} \right\} \sum_{w=1}^{t-1} (-k)^{t-w} \sum_{j=1}^w (-1)^j B_j \left[\begin{matrix} w \\ j \end{matrix} \right].$$

Here $\psi(x)$ is the digamma function, $\left[\begin{matrix} w \\ j \end{matrix} \right]$ is Stirling numbers of the first kind, $\left\{ \begin{matrix} w \\ j \end{matrix} \right\}$ is Stirling numbers of the second kind and B_j is the Bernoulli numbers. Definitions can be found in [1], [2]. See also [4], [5].

2 Proof of Theorem 1.1

Let $u \geq 0$ be real and $r \geq 0$ be integer. For each real $q > -1$ by definition, put

$$f_q(u) = \binom{q}{r} \frac{1}{\Gamma(q+1)} \int_0^\infty x^{q-1} e^{-x} \ln(xu+1) dx \quad (1)$$

where $\Gamma(q+1)$ is the Gamma function. (see for example [2].)

In order to prove Theorem 1.1 we need

Lemma 2.1. *For each real $u \geq 0$ and $\varepsilon \in (-1; -1/2)$ we have*

$$\lim_{m \rightarrow \infty} \sum_{j=0}^m \binom{m}{j} (-1)^j f_{\varepsilon+j}(u) = 0.$$

For each real $u_0 > 0$ the limit converges uniformly for $u \in [0; u_0]$ and $\varepsilon \in (-1; -1/2)$.

Lemma 2.1 will be proved below.

Proof of Theorem 1.1. The proof is in two steps.

Step 1. Let us prove that $\lim_{\varepsilon \rightarrow -1} f_\varepsilon(u) = (-1)^r u$.

We have

$$(-1)^r \lim_{\varepsilon \rightarrow -1} f_\varepsilon(u) = (-1)^r \lim_{\varepsilon \rightarrow -1} \binom{\varepsilon}{r} \frac{1}{\Gamma(1+\varepsilon)} \int_0^\infty x^{\varepsilon-1} \ln(xu+1) e^{-x} dx \stackrel{(*)}{=}$$

$$\stackrel{(*)}{=} \lim_{\varepsilon \rightarrow -1} \frac{1}{\Gamma(1+\varepsilon)} \int_0^\infty x^{\varepsilon-1} \ln(xu+1) e^{-x} dx \stackrel{(**)}{=}$$

$$\stackrel{(**)}{=} \lim_{\varepsilon \rightarrow -1} \frac{1}{\Gamma(1+\varepsilon)} \int_0^\infty x^{\varepsilon-1} (xu) e^{-x} dx = u \lim_{\varepsilon \rightarrow -1} \frac{1}{\Gamma(1+\varepsilon)} \Gamma(1+\varepsilon) = u.$$

The equality (*) follows because

$$\lim_{\varepsilon \rightarrow -1} \binom{\varepsilon}{r} = \binom{-1}{r} = \frac{(-1)(-2) \dots (-r)}{r!} = (-1)^r.$$

The equality (**) follows because

$$\frac{1}{\Gamma(1+\varepsilon)} = \frac{1+\varepsilon}{\Gamma(2+\varepsilon)}.$$

and

$$\begin{aligned} & \left| \int_0^\infty x^{\varepsilon-1} (\ln(xu+1) - xu) e^{-x} dx \right| \leq \\ & \leq \left| \int_0^1 x^{-2} (\ln(xu+1) - xu) e^{-x} \right| + \left| \int_1^\infty x^{-3/2} (\ln(xu+1) - xu) e^{-x} \right|. \end{aligned}$$

Step 2. By Lemma 2.1, we get

$$0 = \lim_{m \rightarrow \infty} \sum_{j=0}^m \binom{m}{j} (-1)^j f_{\varepsilon+j}(u) = f_\varepsilon(u) + \lim_{m \rightarrow \infty} \sum_{j=0}^{m-1} (-1)^{j+1} \binom{m}{j+1} f_{j+1+\varepsilon}(u).$$

In Step 1 we proved that $f_\varepsilon(u)$ tends to $(-1)^r u$ as ε tends to -1 . Hence

$$u = \lim_{\varepsilon \rightarrow -1} \lim_{m \rightarrow \infty} \sum_{j=0}^{m-1} (-1)^{j+r} \binom{m}{j+1} f_{j+1+\varepsilon}(u).$$

Also by Lemma 2.1 the convergence in this formula is uniform for $\varepsilon \in (-1; -1/2)$. Hence changing the order of limits and substituting $m+1$ by m in this formula, we get

$$\begin{aligned} u &= \lim_{m \rightarrow \infty} \sum_{j=0}^m (-1)^{j+r} \binom{m+1}{j+1} f_j(u) = \\ &= \lim_{m \rightarrow \infty} \sum_{j=0}^m \binom{m+1}{j+1} \binom{j}{r} \frac{(-1)^{j+r}}{j!} \int_0^\infty x^{j-1} \ln(xu+1) e^{-x} dx. \end{aligned}$$

Denote by S_m the expression under the limit. We have

$$S_m - S_{m-1} = \sum_{j=0}^m \left(\binom{m+1}{j+1} - \binom{m}{j+1} \right) \binom{j}{r} \frac{(-1)^{j+r}}{j!} \int_0^\infty x^{j-1} \ln(xu+1) e^{-x} dx =$$

$$= \sum_{j=0}^m \binom{m}{j} \binom{j}{r} \frac{(-1)^{j+r}}{j!} \int_0^\infty x^{j-1} \ln(xu+1) e^{-x} dx.$$

Hence

$$u = \sum_{m=0}^\infty (S_m - S_{m-1}) = \sum_{m=0}^\infty \sum_{j=0}^m \binom{m}{j} \binom{j}{r} \frac{(-1)^{j+r}}{j!} \int_0^\infty x^{j-1} \ln(xu+1) e^{-x} dx.$$

□

Proof of Lemma 2.1. Proof is in three steps.

Step 1. Let $j > 0$ be integer and $\varepsilon \in (-1; -1/2)$ be real. We claim that

$$f_{\varepsilon+j}(u) = \binom{\varepsilon+j-r}{j}^{-1} \sum_{i=0}^j \binom{\varepsilon+j-1}{j-i} \frac{u^i}{i!} f_\varepsilon^{(i)}(u). \quad (2)$$

The proof is by induction over j . Let us prove the base of induction for $j = 1$. We must prove that

$$f_{\varepsilon+1}(u) = \frac{\varepsilon}{\varepsilon+1-r} f_\varepsilon(u) + \frac{1}{\varepsilon+1-r} u f'_\varepsilon(u). \quad (3)$$

Integrating formula (1) by part, we get

$$\begin{aligned} f_\varepsilon(u) &= -\binom{\varepsilon}{r} \frac{1}{\Gamma(\varepsilon+1)} \int_0^\infty \frac{x^\varepsilon}{\varepsilon} \left(-e^{-x} \ln(xu+1) + u \frac{e^{-x}}{xu+1} \right) dx = \\ &= \frac{\varepsilon-r+1}{\varepsilon} f_{\varepsilon+1}(u) - \frac{u}{\varepsilon} \binom{\varepsilon}{r} \frac{1}{\Gamma(\varepsilon+1)} \int_0^\infty x^\varepsilon e^{-x} \frac{dx}{xu+1}. \end{aligned} \quad (4)$$

For each real $u_0 > 0$ integral in formula (1) converges uniformly for $u \in [0; u_0]$. Hence differentiating formula (1) with respect to u , we get

$$f'_\varepsilon(u) = \binom{\varepsilon}{r} \frac{1}{\Gamma(\varepsilon + 1)} \int_0^\infty x^\varepsilon e^{-x} \frac{dx}{xu + 1}.$$

Combining this with formula (4), we obtain

$$f_\varepsilon(u) = \frac{\varepsilon - r + 1}{\varepsilon} f_{\varepsilon+1}(u) - \frac{u}{\varepsilon} f'_\varepsilon(u).$$

The base of induction follows.

Let us prove the step of induction. By the inductive hypothesis for $j = N$, substituting $\varepsilon + 1$ for ε , we get

$$f_{\varepsilon+N+1}(u) = \binom{\varepsilon + N + 1 - r}{N}^{-1} \sum_{i=0}^N \binom{\varepsilon + N}{N - i} \frac{u^i}{i!} f_{\varepsilon+1}^{(i)}(u).$$

Substituting formula (3) in this formula, we get

$$\begin{aligned} f_{\varepsilon+N+1}(u) &= \\ &= \binom{\varepsilon + N + 1 - r}{N}^{-1} \sum_{i=0}^N \binom{\varepsilon + N}{N - i} \frac{u^i}{i!} \frac{d^i}{du^i} \left(\frac{\varepsilon}{\varepsilon + 1 - r} f_\varepsilon(u) + \frac{1}{\varepsilon + 1 - r} u f'_\varepsilon(u) \right). \end{aligned}$$

Or equivalently

$$f_{\varepsilon+N+1}(u) (N+1) \binom{\varepsilon + N + 1 - r}{N + 1} = \sum_{i=0}^N \binom{\varepsilon + N}{N - i} \frac{u^i}{i!} \frac{d^i}{du^i} (\varepsilon f_\varepsilon(u) + u f'_\varepsilon(u)). \quad (5)$$

If we substituting in the Leibniz formula

$$\frac{d^i}{du^i} (h(u)g(u)) = \sum_{k=0}^i \binom{i}{k} f^{(k)}(u) g^{(i-k)}(u)$$

u for $h(u)$ and $f'(u)$ for $g(u)$, we obtain

$$\frac{d^i}{du^i}(uf'_\varepsilon(u)) = uf_\varepsilon^{(i+1)}(u) + if_\varepsilon^{(i)}(u).$$

Hence the right-hand side of formula (5) can be rewritten as

$$\begin{aligned} & \sum_{i=0}^N \binom{\varepsilon + N}{N-i} \frac{u^i}{i!} (\varepsilon f_\varepsilon^{(i)}(u) + uf_\varepsilon^{(i+1)}(u) + if_\varepsilon^{(i)}(u)) = \\ &= \sum_{i=0}^N \binom{\varepsilon + N}{N-i} \frac{u^i}{i!} (\varepsilon + i) f_\varepsilon^{(i)}(u) + \sum_{i=0}^N \binom{\varepsilon + N}{N-i} \frac{u^i}{i!} uf_\varepsilon^{(i+1)}(u) = \\ &= \sum_{i=0}^{N+1} \frac{u^i}{i!} f_\varepsilon^i(u) \left(\binom{\varepsilon + N}{N-i} (\varepsilon + i) + \binom{\varepsilon + N}{N-i+1} i \right). \end{aligned}$$

From the formula

$$\binom{\varepsilon + N}{N-i} (\varepsilon + i) + \binom{\varepsilon + N}{N-i+1} i = (N+1) \binom{\varepsilon + N}{N-i+1}$$

it follows that

$$f_{\varepsilon+N+1}(u)(n+1) \binom{\varepsilon + N + 1 - r}{N+1} = (n+1) \sum_{i=0}^{N+1} \frac{u^i}{i!} f_\varepsilon^i(u) \binom{\varepsilon + N}{N-i+1}.$$

Dividing both sides by $(n+1) \binom{\varepsilon+N+1-r}{N+1}$, we get formula (2) for $j = N+1$. The step of induction follows.

Step 2. Let us prove that

$$\sum_{j=i}^m \binom{m}{j} \binom{\varepsilon + j - r}{j}^{-1} \binom{\varepsilon + j - 1}{j-i} (-1)^j = \binom{m-i-r}{m-i} \binom{m+\varepsilon-r}{m}^{-1} (-1)^i. \quad (6)$$

By definition, put

$$F(a, b, c; x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \frac{a(a+1) \dots (a+k-1)b(b+1) \dots (b+k-1)}{c(c+1) \dots (c+k-1)}.$$

This series converges, if $|x| \leq 1$ and $a + b < c$.

We have

$$\begin{aligned} & \sum_{j=i}^m \binom{m}{j} \binom{\varepsilon + j - r}{j}^{-1} \binom{\varepsilon + j - 1}{j - i} (-x)^j = \\ &= \frac{m! \Gamma(\varepsilon - r + 1)}{(m - i)! \Gamma(\varepsilon - r + i + 1)} (-x)^i F(i + \varepsilon, i - m, \varepsilon + i - r + 1; x). \end{aligned}$$

Let us prove that

$$\begin{aligned} & \frac{m! \Gamma(\varepsilon - r + 1)}{(m - i)! \Gamma(\varepsilon - r + i + 1)} F(i + \varepsilon, i - m, \varepsilon + i - r + 1; 1) = \\ &= \binom{m - i - r}{m - i} \binom{m + \varepsilon - r}{m}^{-1} = \frac{(m - i - r)! m! \Gamma(\varepsilon - r + 1)}{(m - i)! \Gamma(1 - r) \Gamma(m + \varepsilon - r + 1)}. \end{aligned}$$

Or equivalently

$$F(i + \varepsilon, i - m, \varepsilon + i - r + 1; 1) = \frac{(m - i - r)! \Gamma(\varepsilon - r + i + 1)}{\Gamma(1 - r) \Gamma(m + \varepsilon - r + 1)}.$$

This formula follows by the Gauss's theorem (see [2, p. 282])

$$F(a, b, c; 1) = \frac{\Gamma(c - a - b) \Gamma(c)}{\Gamma(c - a) \Gamma(c - b)},$$

for $a = i + \varepsilon, b = i - m$ and $c = \varepsilon + i - r + 1$.

Formula (6) is proved.

Step 3. From formula (2) it follows that

$$\begin{aligned}
& \sum_{j=0}^m \binom{m}{j} (-1)^j f_{\varepsilon+j}(u) = \\
&= \sum_{j=0}^m \binom{m}{j} (-1)^j \binom{\varepsilon+j-r}{j}^{-1} \sum_{i=0}^j \binom{\varepsilon+j-1}{j-i} \frac{u^i}{i!} f_{\varepsilon}^{(i)}(u) = \\
&= \sum_{j=0}^m \sum_{i=0}^j \binom{m}{j} \binom{\varepsilon+j-r}{j}^{-1} \binom{\varepsilon+j-1}{j-i} (-1)^j \frac{u^i}{i!} f_{\varepsilon}^{(i)}(u) = \\
&= \sum_{i=0}^m \sum_{j=i}^m \binom{m}{j} \binom{\varepsilon+j-r}{j}^{-1} \binom{\varepsilon+j-1}{j-i} (-1)^j \frac{u^i}{i!} f_{\varepsilon}^{(i)}(u) = \\
&= \sum_{i=0}^m \frac{u^i}{i!} f_{\varepsilon}^{(i)}(u) \sum_{j=i}^m \binom{m}{j} \binom{\varepsilon+j-r}{j}^{-1} \binom{\varepsilon+j-1}{j-i} (-1)^j.
\end{aligned}$$

Combining this with formula (6), we obtain

$$\binom{m+\varepsilon-r}{m}^{-1} \sum_{i=0}^m \frac{u^i}{i!} f_{\varepsilon}^{(i)}(u) (-1)^i \binom{m-i-r}{m-i}. \quad (7)$$

Step 4. In this step we need

Lemma 2.2. *Let $u \geq 0$ and $\varepsilon \in (-1; -1/2)$ be real and $n > 0$ be integer. We have*

$$\lim_{m \rightarrow \infty} m^n \frac{f_{\varepsilon}^{(m)}(u) u^m}{u(1+\varepsilon)m!} = 0.$$

For each real $u_0 > 0$ the limit converges uniformly for $u \in [0; u_0]$ and $\varepsilon \in (-1; -1/2)$.

Lemma 2.2 will be proved below.

Let us consider two cases.

Case 1 : let r be zero. Let x and θ be real and $0 \leq x \leq u, \theta \in (0; 1)$.

By the Taylor's theorem in the Cauchy form for the function $f_\varepsilon(u)$, we obtain

$$f_\varepsilon(x) = \sum_{i=0}^m \frac{(x-u)^i}{i!} f_\varepsilon^{(i)}(u) + \frac{(x-u)^{m+1}(1-\theta)^m}{m!} f_\varepsilon^{m+1}(u+\theta(x-u)).$$

Putting in this formula $x = 0$, we obtain

$$\sum_{i=0}^m \frac{u^i}{i!} f_\varepsilon^{(i)}(u)(-1)^i = \frac{u^{m+1}(-1)^m(1-\theta)^m}{m!} f_\varepsilon^{m+1}(u(1-\theta)).$$

Hence using inequality

$$\begin{aligned} \binom{m+\varepsilon}{m}^{-1} &= \frac{m!}{(\varepsilon+1)\dots(\varepsilon+m)} = \\ &= \frac{m}{1+\varepsilon} \left(\frac{m-1}{\varepsilon+m} \right) \dots \left(\frac{1}{\varepsilon+2} \right) < \frac{m}{1+\varepsilon} \end{aligned}$$

we have

$$\begin{aligned} &\left| \binom{m+\varepsilon}{m}^{-1} \sum_{i=0}^m \frac{u^i}{i!} f_\varepsilon^{(i)}(u)(-1)^i \right| < \\ &< m(m+1)(-1)^m u \frac{f_\varepsilon^{m+1}(u(1-\theta))(u(1-\theta))^{m+1}}{(u(1-\theta))(1+\varepsilon)(m+1)!}. \end{aligned}$$

The left-hand side of this inequality equals expression (7), but for each real $u_0 > 0$ the right-hand side of this inequality tends to 0 as m tends to ∞ uniformly for $u \in [0; u_0]$ and $\varepsilon \in (-1; -1/2)$ by Lemma 2.2.

Case 2 : let r be positive. Expression (7) can be rewritten as

$$\binom{m+\varepsilon-r}{m}^{-1} \sum_{i=m-r+1}^m \frac{u^i}{i!} f_\varepsilon^{(i)}(u)(-1)^i \binom{m-i-r}{m-i}.$$

Because for $m - i - r \geq 0$, we get $\binom{m-i-r}{m-i} = 0$. Let $j \in [0; r - 1]$ be integer. We must prove that

$$\lim_{m \rightarrow \infty} \binom{m + \varepsilon - r}{m}^{-1} \frac{u^{m-j}}{(m-j)!} f_{\varepsilon}^{(m-j)}(u) (-1)^{m-j} \binom{j-r}{j} = 0.$$

Or equivalently

$$\lim_{m \rightarrow \infty} \binom{m + j + \varepsilon - r}{m + j}^{-1} \frac{u^m}{m!} f_{\varepsilon}^{(m)}(u) = 0.$$

There exists integer number n and real number C such that $\left| \binom{m+j+\varepsilon-r}{m+j}^{-1} \right| < C m^n$. Hence

$$\left| \binom{m + j + \varepsilon - r}{m + j}^{-1} \frac{u^m}{m!} f_{\varepsilon}^{(m)}(u) \right| < C m^n \frac{u^m}{m!} f_{\varepsilon}^{(m)}(u).$$

By Lemma 2.2 the right-hand of this inequality tends to 0 as m tends to ∞ .

□

Proof of Lemma 2.2. For each real $u_0 > 0$ integral in formula (1) converges uniformly for $u \in [0; u_0]$. Hence differentiating formula (1) with respect to u , we get

$$f_{\varepsilon}^{(m)}(u) u^m = (-1)^{m+1} (m-1)! \binom{\varepsilon}{r} \frac{1}{\Gamma(1+\varepsilon)} \int_0^{\infty} x^{\varepsilon-1} e^{-x} \frac{(xu)^m}{(xu+1)^m} dx.$$

Let $T = \sqrt{m}$ and $m > 4$. We have

$$\begin{aligned} \int_0^{\infty} x^{\varepsilon-1} e^{-x} \frac{(xu)^m}{(xu+1)^m} dx = \\ \int_0^T x^{\varepsilon-1} e^{-x} \left(1 + \frac{1}{xu}\right)^{-m} dx + \int_T^{\infty} x^{\varepsilon-1} e^{-x} \left(1 + \frac{1}{xu}\right)^{-m} dx. \end{aligned} \quad (8)$$

The first term. For each real $x \in [0; T]$, we have

$$\int_0^T x^{\varepsilon-1} e^{-x} \left(1 + \frac{1}{xu}\right)^{-m} dx < u^2 T^{2+\varepsilon} \left(1 + \frac{1}{Tu}\right)^{2-m}$$

because

$$x^{\varepsilon-1} \left(1 + \frac{1}{xu}\right)^{-m} \leq u^2 T^{1+\varepsilon} \left(1 + \frac{1}{Tu}\right)^{2-m} \quad \text{and} \quad e^{-x} \leq 1.$$

The second term. For each real $x \in [T; +\infty)$, we have

$$\int_T^\infty x^{\varepsilon-1} e^{-x} \left(1 + \frac{1}{xu}\right)^{-m} dx < \int_T^\infty e^{-x} (xu) dx = u e^{-T} (T + 1)$$

because

$$x^{\varepsilon-1} \left(1 + \frac{1}{xu}\right)^{-m+1} \frac{xu}{1+xu} < xu.$$

Hence

$$\left| m^n \frac{f_\varepsilon^{(m)}(u) u^m}{u(1+\varepsilon)m!} \right| < \frac{(-1)^{m+1} m^{n-1}}{\Gamma(2+\varepsilon)} \binom{\varepsilon}{r} \left(u \sqrt{m}^{2+\varepsilon} \left(1 + \frac{1}{\sqrt{mu}}\right)^{2-m} + e^{-\sqrt{m}} (\sqrt{m} + 1) \right).$$

Clearly, for each real $u_0 > 0$ the expression in the right-hand sides tends to 0 as m tends to ∞ uniformly for $u \in [0; u_0]$ and $\varepsilon \in (-1; -1/2)$.

□

3 Proof of Corollary 1.2 and Corollary 1.3

In order to prove Corollary 1.2 and Corollary 1.3 we need

Lemma 3.1. *For each integer $n \geq 0$*

$$\int_0^\infty \frac{x^n}{x+1} e^{-x} dx = (-1)^n \left(\sum_{j=0}^{n-1} (j!(-1)^{j+1}) + \delta \right) \quad (9)$$

and

$$\int_0^\infty x^n \ln(x+1) e^{-x} dx = \sum_{j=0}^n \frac{n!}{j!} (-1)^j \left(\sum_{i=0}^{j-1} (i!(-1)^{i+1}) + \delta \right). \quad (10)$$

Formula (9) can be found in [3, f. 3.353.5], formula (10) can be found in [3, f. 4.337.5].

Proof of Corollary 1.2. The formula of Theorem 1.1 converges uniformly for $u \in [0; 1]$. Hence differentiating the formula of Theorem 1.1 respect with to u and taking $u = 1$, we get

$$1 = \sum_{m=r}^{\infty} \sum_{k=r}^m \binom{m}{k} \binom{k}{r} \frac{(-1)^{k+r}}{k!} \int_0^{\infty} \frac{x^k e^{-x}}{x+1} dx.$$

Series in the right-hand side of this formula converges. Hence

$$\lim_{m \rightarrow \infty} \sum_{k=r}^m \binom{m}{k} \binom{k}{r} \frac{(-1)^{k+r}}{k!} \int_0^{\infty} \frac{x^k e^{-x}}{x+1} dx = 0.$$

By formula (9) of Lemma 3.1, we get

$$\begin{aligned} & \sum_{k=r}^m \binom{m}{k} \binom{k}{r} \frac{(-1)^{k+r}}{k!} \int_0^{\infty} \frac{x^k e^{-x}}{x+1} dx = \\ &= \sum_{k=r}^m \binom{m}{k} \binom{k}{r} \frac{(-1)^{k+r}}{k!} (-1)^k \left(\sum_{j=0}^{k-1} (j! (-1)^{j+1}) + \delta \right) = \\ & (-1)^r \sum_{k=r}^m \sum_{j=0}^{k-1} \binom{m}{k} \binom{k}{r} \frac{j!}{k!} (-1)^{j+1} + (-1)^r \sum_{k=r}^m \binom{m}{k} \binom{k}{r} \frac{1}{k!} \delta = \\ &= \frac{(-1)^r}{m!} \sum_{k=r}^m \binom{m}{k}^2 \binom{k}{r} (m-k)! \sum_{j=0}^{k-1} j! (-1)^{j+1} + \delta \frac{(-1)^r}{m!} \sum_{k=r}^m \binom{m}{k}^2 \binom{k}{r} (m-k)!. \end{aligned}$$

Clearly, the expression

$$\sum_{k=r}^m \binom{m}{k} \binom{k}{r} \frac{1}{k!}$$

tends to ∞ as m tends to ∞ . Hence

$$\lim_{m \rightarrow \infty} \frac{\sum_{k=r}^m \binom{m}{k}^2 \binom{k}{r} (m-k)! \sum_{j=0}^{k-1} j! (-1)^{j+1}}{\sum_{k=r}^m \binom{m}{k}^2 \binom{k}{r} (m-k)!} = -\delta.$$

□

Proof of Corollary 1.3. Series in the right-hand side of the formula of Theorem 1.1 converges. Hence

$$\lim_{m \rightarrow \infty} \left(\sum_{k=r}^m \binom{m}{k} \binom{k}{r} \frac{(-1)^{k+r}}{k!} \int_0^\infty x^{k-1} e^{-x} \ln(xu+1) dx \right) = 0.$$

Taking $u = 1$ in this formula and using formula (10) of Lemma 3.1, we obtain

$$\lim_{m \rightarrow \infty} \left[\sum_{k=r}^m \binom{m}{k} \binom{k}{r} \frac{(-1)^{k+r}}{k!} \sum_{j=0}^{k-1} \frac{(k-1)!}{j!} (-1)^j \left(\sum_{i=0}^{j-1} (i! (-1)^{i+1}) + \delta \right) \right] = 0.$$

Or equivalently

$$\sum_{k=r}^m \sum_{j=0}^{k-1} \sum_{i=0}^{j-1} \binom{m}{k} \binom{k}{r} \frac{i!}{kj!} (-1)^{k+r+j+i+1} + \delta \sum_{k=r}^m \sum_{j=0}^{k-1} \binom{m}{k} \binom{k}{r} \frac{(-1)^{k+r+j}}{kj!}.$$

We must prove that the expression

$$A_m(r) := r(-1)^{r+1} \sum_{k=r}^m \sum_{j=0}^{k-1} \binom{m}{k} \binom{k}{r} \frac{(-1)^{k+r+j}}{kj!} = \sum_{j=1}^m \frac{(-1)^j}{(j-1)!} \sum_{k=j}^m \binom{m}{k} \binom{k-1}{r-1} (-1)^i$$

tends to $+\infty$ as m tends to ∞ . We claim that $A_m(r+2) - A_m(r)$ tends to $+\infty$ as m tends to ∞ . We have

$$A_m(r) + A_m(r+1) = \sum_{j=1}^m \frac{(-1)^j}{(j-1)!} \sum_{k=j}^m \binom{m}{k} \left(\binom{k-1}{r-1} + \binom{k-1}{r} \right) (-1)^k =$$

$$= \sum_{j=1}^m \frac{(-1)^j}{(j-1)!} \sum_{k=j}^m \binom{m}{k} \binom{k}{r} (-1)^k = \sum_{j=1}^m \binom{m}{j} \binom{j}{r} \frac{j-r}{m-r} \frac{1}{(j-1)!}$$

because

$$\sum_{k=j}^m \binom{m}{k} \binom{k}{r} (-1)^k = \binom{m}{j} \binom{j}{r} \frac{j-r}{m-r} (-1)^j. \quad (11)$$

This formula will be proved below.

Hence for $m > r + 1$, we obtain

$$\begin{aligned} A_m(r+2) - A_m(r) &= (A_m(r+1) + A_m(r+2)) - (A_m(r) + A_m(r+1)) = \\ &= \sum_{j=1}^m \binom{m}{j} \binom{j}{r} \frac{j-r}{(j-1)!} \left(\frac{j-r-1}{(m-r-1)(r+1)} - \frac{1}{m-r} \right) > \\ &> \sum_{j=r+1}^{2r+3} \binom{m}{j} \binom{j}{r} \frac{j-r}{(j-1)!} \left(\frac{j-r-1}{(m-r-1)(r+1)} - \frac{1}{m-r} \right) = \frac{P(m)}{m-r-1} \end{aligned}$$

Here $P(m)$ is a polynomial, $\deg(P) \geq 2r+2 \geq 2$. Hence the right-hand sides tends to $+\infty$ as m tends to ∞ . But $A_m(0) = 0$ and

$$A_m(1) = A_m(0) + A_m(1) = \sum_{j=1}^m \binom{m-1}{j-1} \frac{1}{(j-1)!}.$$

Hence $A_m(r)$ tends to $+\infty$ as m tends to ∞ for each positive integer r .

Let us prove formula 11. We have

$$\begin{aligned} \sum_{k=j}^m \binom{m}{k} \binom{k}{r} (-1)^k &= (-x)^j \frac{m!}{(m-j)!r!(j-r)!} F(1, j-m, 1+j-r; x) = \\ &= (-x)^j \binom{m}{j} \binom{j}{r} F(1, j-m, 1+j-r; x). \end{aligned}$$

Hence we must prove that

$$F(1, j - m, 1 + j - r; 1) = \frac{j - r}{m - r}.$$

This formula follows by the Gauss theorem

$$F(a, b, c; 1) = \frac{\Gamma(c - a - b)\Gamma(c)}{\Gamma(c - a)\Gamma(c - b)}$$

for $a = 1, b = j - m, c = 1 + j - r$.

□

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